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Soliton equations solved by the boundary CFT

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Abstract: Soliton equations are derived which characterize the boundary CFT *à la* Callan et al. Soliton fields of classical soliton equations are shown to appear as a neutral bound state of a pair of soliton fields of BCFT. One soliton amplitude under the influence of the boundary is calculated explicitly and is shown that it is frozen at the Dirichlet limit.

Keywords: Conformal Field Models in String Theory, Conformal and W Symmetry, Integrable Hierarchies, Integrable Equations in Physics.

Contents

1. Introduction

An interesting exact solution of the boundary CFT was found by Callan et al. in [1]. The solution, which minimizes the boundary action, admits a soliton to propagate from the boundary.

On the other hand it has been shown in [2] that an arbitrary open string correlation function satisfies the Hirota bilinear difference equation (HBDE) [3, 4], a discrete soliton equation which generates equations of the KP hierarchy in various continuous limits. The purpose of this note is to point out that the boundary CFT solution of [1] satisfies a simple generalization of HBDE and clarify the correspondence of solitons in two theories.

Let us first recall the exact solution of the boundary CFT derived by Callan et al. [1]. They consider a scalar field $X(z)$ which has a periodicity $X \sim X + 2\pi\sqrt{2}$ on the boundary. The SU(2) currents are introduced by

$$J^{\pm}(z) =: e^{\pm i\sqrt{2}X(z)} :, \qquad J^3(z) = \frac{i}{\sqrt{2}}\frac{\partial X(z)}{\partial z}$$

and the interacting boundary state is obtained from the Neumann boundary state $|N\rangle$ by an SU(2) rotation according to the following formula

$$|B\rangle = e^{i\theta_a J^a}|N\rangle, \qquad \theta_a J^a = \pi(gJ^+ + \bar{g}J^-)\,, \tag{1.1}$$

where g is a complex coupling constant and

$$J^{\pm} = \oint \frac{dz}{2\pi i} J^{\pm}(z)\,, \qquad J^3 = \oint \frac{dz}{2\pi i} J^3(z)\,.$$

Under this circumstance $1 \to 1$ particle correlation function is calculated as

$$\langle \partial X(z,\bar{z})\bar{\partial}X(z',\bar{z}')\rangle = -\frac{\cos 2\pi g}{(z-\bar{z}')^2}\,.$$

The rest of the probability is scattered into charge sectors according to

$$\langle \bar{\partial}X(\bar{z}')e^{\pm i\sqrt{2}X(z)}\rangle = \pm\frac{1}{\sqrt{2}}\frac{\sin 2\pi g}{(z-\bar{z}')^2}\,,$$

which were interpreted as solitons propagating away from the boundary. Since the solitons, which appear in the boundary CFT, are topological ones, it seems, at first glance, nothing to do with solutions of classical soliton equations, such as the KdV equation, the Toda lattice, the sine-Gordon equation and so on.

Despite of this natural guess we would like to show, in this paper, that the BCFT solitons and those of the classical equations share a common origin. We will derive a soliton equation which is satisfied by the boundary CFT correlation functions. A soliton solution to the equation turns out to be a neutral bound state of soliton fields of the boundary CFT.

We review briefly the string-soliton correspondence in the case of open strings in section 2, so that solutions to the classical soliton equations are expressed in terms of the language of the open string theory. In section 3 the string-soliton correspondence is extended to the closed string theory. The manifestly symmetric conformal bound states are reexamined within our formulation in section 4 and the soliton equation, which characterizes the BCFT, is derived in section 5. The 'soliton fields' in two theories will be compared and their relations will be examined in section 6. In the final section we will calculate explicitly one soliton solution under the influence of the boundary.

2. String-soliton correspondence in the case of open strings

In order to clarify our assertion it will be most instructive to study the structure of one soliton solution written in the form of τ function:

$$\tau_{1\mathrm{sol}} = 1 + \frac{a}{z-z'}e^{\xi(t,z)-\xi(t,z')}\,, \tag{2.1}$$

$$\xi(t,z) := -\sum_{n=0}^{\infty} t_n z^n\,. \tag{2.2}$$

To be specific let us consider the case of Toda lattice as an example. If we put

$$t_1 = m\,, \qquad t_3 = t\,, \qquad t_j = 0 \qquad (j \neq 1,3)$$

the function $r_m(t)$ defined by

$$e^{-r_m} = 1 + \frac{d^2}{dt^2}\ln\tau(t,m) \tag{2.3}$$

satisfies the Toda lattice equation

$$\frac{d^2 r_m}{dt^2} = 2e^{-r_m} - e^{-r_{m+1}} - e^{-r_{m-1}}\,.$$

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The complex parameters z, z' determine the properties of the soliton, i.e.,

$$
\begin{aligned}
e^{-r_m} - 1 &= \omega^2 \ \cosh^{-2}(\kappa m + \omega t + \delta)\,, \\
\kappa = \frac{1}{2}(z' - z)\,, \qquad \omega &= \frac{1}{2}\left(z'^3 - z^3\right), \qquad e^{2\delta} = \frac{a}{z - z'}\,.
\end{aligned} \tag{2.4}
$$

Now we show that the τ function of (2.1) can be described in terms of the string theory. For this purpose we introduce free fermion fields $\psi(z)$ and $\psi^*(z)$ satisfying the anti-commutation relations

$$
\begin{aligned}
\{\psi(z),\ \psi(z')\} &= \{\psi^*(z),\ \psi^*(z')\} = 0\,, \\
\{\psi(z),\ \psi^*(z')\} &= 2\pi i \delta(z - z')
\end{aligned} \tag{2.5}
$$

and write τ_{1sol} as

$$
\tau_{1\mathrm{sol}} = \langle 0| e^{H(t)} e^{a\psi(z)\psi^*(z')} |0\rangle\,, \tag{2.6}
$$

where

$$
H(t) = \sum_{n=0}^{\infty} t_n \oint \frac{dz}{2\pi i} z^n \psi^*(z)\psi(z)\,. \tag{2.7}
$$

The vacuum state $|0\rangle$ is assumed to satisfy

$$
H(t)|0\rangle = 0\,.
$$

The Grassmann nature of the fields $\psi,\ \psi^*$ and the properties

$$
\begin{aligned}
e^{H(t)}\psi(z)e^{-H(t)} &= e^{\xi(t,z)}\psi(z)\,, \\
e^{H(t)}\psi^*(z)e^{-H(t)} &= e^{-\xi(t,z)}\psi^*(z)
\end{aligned}
$$

will show that (2.6) reproduces the τ of (2.1).

One of the keys to the correspondence between the soliton theory and the string theory is the bosonization of the fields ψ. It is done by writing $\psi(z)$ in terms of the string coordinate $X^\mu(z)$:

$$
\begin{aligned}
X^\mu(z) &= X^\mu_+(z) + X^\mu_-(z) \\
X^\mu_+(z) &= x^\mu_0 - i\sum_{n=1}^{\infty} \frac{1}{n}\alpha^\mu_{-n} z^n\,, \\
X^\mu_-(z) &= -ip^\mu \ln z + i\sum_{n=1}^{\infty} \frac{1}{n}\alpha^\mu_n z^{-n}\,.
\end{aligned} \tag{2.8}
$$

We consider only one dimensional space time, for simplicity, hence do not write the space-time index μ hereafter. Using the standard relations

$$
[x, p] = i\,, \qquad [\alpha_m,\ \alpha_n] = m\delta_{m+n}\,, \qquad m, n \neq 0 \tag{2.9}
$$

we can show that

$$
\begin{aligned}
\psi(z) &= \ :e^{iX(z)}: \equiv e^{iX_+(z)} e^{iX_-(z)}\,, \\
\psi^*(z) &= \ :e^{-iX(z)}:
\end{aligned}
$$

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satisfy the anti-commutation relations (2.5). Correspondingly the vacuum state $|0\rangle$ is annihilated by p and $\alpha_n,\ n<0$. The hamiltonian $H(t)$ of (2.7) is now given by

$$H(t) = \oint \frac{dz}{2\pi}\xi(t,z)\frac{\partial X(z)}{\partial z}\,. \tag{2.10}$$

Another key to the correspondence is the Miwa transformation of the variables:

$$t_0 = -\sum_{j=1}^{M} k_j \ln z_j\,, \qquad t_n = \frac{1}{n}\sum_{j=1}^{M} k_j z_j^{-n}\,. \tag{2.11}$$

The maximum number M of the summation can be chosen arbitrarily. The set of new variables $\{k_j, z_j\}$ enable us to describe the hamiltonian and the function $\tau_{1\mathrm{sol}}$ fully by the language of the string theory. In fact an equivalent expression of $\xi(t,z)$ is given by

$$\xi(t,z) = \sum_{j=1}^{M} k_j \Delta_-(z_j, z)\,.$$

Here by $\Delta_\pm(z,z')$ we denote the $2D$ Green's function $\ln|z-z'|$ corresponding to the following power series expansions

$$\Delta_\pm(z,z') := \begin{pmatrix} \ln|z'| \\ \ln|z| \end{pmatrix} - \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{z}{z'}\right)^{\pm n}\,. \tag{2.12}$$

Accordingly we have the string representations of the hamiltonian and the $\tau_{1\mathrm{sol}}$ as

$$\begin{aligned} H(t) &= i\sum_{j=1}^{M} k_j X_-(z_j), && (2.13)\\ \tau_{1\mathrm{sol}} &= \langle 0| : e^{i\sum_j k_j X(z_j)} : \exp\left[a : e^{iX(z)} :: e^{-iX(z')} :\right] |0\rangle && (2.14)\\ &= \langle 0| : e^{i\sum_j k_j X(z_j)} : \left(1 + a : e^{iX(z)} :: e^{-iX(z')} :\right) |0\rangle\,. && (2.15) \end{aligned}$$

General solutions to the soliton equations belonging to the KP hierarchy are given, in terms of fermion fields, in the form

$$\tau(t) = \langle 0|e^{H(t)}G|0\rangle\,. \tag{2.16}$$

Here G is an element of the universal grassmannian introduced by M.Sato[5]. In the papers [2, 6] one of the present authors showed that (2.16) can be rewritten by using string coordinates and satisfies explicitly the Hirota bilinear difference equation[3, 4]. Namely we can write (2.16) as

$$\begin{aligned} \tau = \langle 0| : e^{i\sum_j k_j X(z_j)} : \theta\left(\zeta - \oint \frac{dX(z)}{2\pi} w(z)\right) \times \\ \times \exp\left[\frac{1}{2}\oint\frac{dX(z)}{2\pi}\oint\frac{dX(z')}{2\pi}\ln\frac{E(z,z')}{z-z'}\right]|0\rangle\,, \end{aligned} \tag{2.17}$$

where w, θ and $E(z,z')$ are the first Abel differential, the Riemann theta function and the prime form, respectively, associated to some Riemann surface[7]. In fact the substitution of (2.17) into the HBDE

$$\begin{aligned} &\frac{\tau(k_1+1,k_2,k_3)\tau(k_1,k_2+1,k_3+1)}{(z_1-z_2)(z_1-z_3)}+ \\ &+\frac{\tau(k_1,k_2+1,k_3)\tau(k_1+1,k_2,k_3+1)}{(z_2-z_1)(z_2-z_3)}+ \\ &+\frac{\tau(k_1,k_2,k_3+1)\tau(k_1+1,k_2+1,k_3)}{(z_3-z_2)(z_3-z_1)} = 0 \end{aligned} \tag{2.18}$$

yields Fay's trisecant formula [8]. This formula is correct for any choice of three k_j's among $\{k_1,k_2,k_3,\ldots,k_M\}$.

3. String-soliton correspondence in the case of closed strings

So far we have used only open strings to represent the τ functions. In order to study the boundary CFT we have to incorporate the closed string coordinate. Namely we introduce, in addition to (2.8), the anti-holomorphic coordinate

$$\begin{aligned} \tilde{X}(\bar{z}) &= \tilde{X}_+(\bar{z})+\tilde{X}_-(\bar{z}), \\ \tilde{X}_+(\bar{z}) &= \tilde{x}_0 - i\sum_{n=1}^{\infty}\frac{1}{n}\tilde{\alpha}_{-n}\bar{z}^n, \\ \tilde{X}_-(\bar{z}) &= -i\tilde{p}\ln\bar{z} + i\sum_{n=1}^{\infty}\frac{1}{n}\tilde{\alpha}_n\bar{z}^{-n}, \end{aligned}$$

whose components are constrained by the same relations as (2.9) and commute with $X(z)$. The closed string coordinate is then given by

$$X(z,\bar{z}) = X(z)+\tilde{X}(\bar{z}).$$

In order to find a fermionic field $\psi(z,\bar{z})$ associated with closed strings, first we notice the following identity:

$$:e^{ikX(z,\bar{z})}::e^{ik'X(z',\bar{z}')}:=(z-z')^{kk'}(\bar{z}-\bar{z}')^{kk'}:e^{ikX(z,\bar{z})+ik'X(z',\bar{z}')}:. \tag{3.1}$$

When $k=-k'=1/\sqrt{2}$ and $z'\neq z,\ \bar{z}'\neq\bar{z}$,

$$\begin{aligned} &\left\{:e^{iX(z,\bar{z})/\sqrt{2}}:,\ :e^{-iX(z',\bar{z}')/\sqrt{2}}:\right\} = \\ &=\left(\frac{1}{(z-z')^{1/2}(\bar{z}-\bar{z}')^{1/2}}+\frac{1}{(z'-z)^{1/2}(\bar{z}'-\bar{z})^{1/2}}\right):e^{i(X(z,\bar{z})-iX(z',\bar{z}'))/\sqrt{2}}: \\ &=0 \end{aligned}$$

holds. At $z'=z,\ \bar{z}'=\bar{z}$ the quantity in the bracket () diverges. Hence we obtain a bosonization for the closed strings

$$\psi(z,\bar{z})=:e^{iX(z,\bar{z})/\sqrt{2}}:,\qquad \psi^*(z,\bar{z})=:e^{-iX(z,\bar{z})/\sqrt{2}}: \tag{3.2}$$

which satisfy

$$\begin{aligned}\{\psi(z,\bar z),\ \psi(z',\bar z')\} &= \{\psi^*(z,\bar z),\ \psi^*(z',\bar z')\} = 0\,,\\ \{\psi(z,\bar z),\ \psi^*(z',\bar z')\} &= (2\pi i)^2\delta(z'-z)\delta(\bar z'-\bar z)\,.\end{aligned} \tag{3.3}$$

To define the vacuum state of the fermionic fields we expand the fields $\psi(z,\bar z)$ and $\psi^*(z,\bar z)$ as

$$\begin{aligned}\psi(z,\bar z) &= \sum_{m,n=-\infty}^{\infty}\psi_{mn}z^m\bar z^n\,,\\ \psi^*(z,\bar z) &= \sum_{m,n=-\infty}^{\infty}\psi^*_{mn}z^{-m-1}\bar z^{-n-1}\,.\end{aligned}$$

Using (3.3) we find

$$\{\psi_{mn},\psi^*_{rs}\} = \delta_{mr}\delta_{ns}\,. \tag{3.4}$$

If we impose for the vacuum state to satisfy

$$\begin{aligned}&\psi_{mn}|0\rangle \neq 0\,, \qquad \langle 0|\psi^*_{mn} \neq 0\,, \qquad \text{iff } m,n\geq 0,\\ &\psi^*_{mn}|0\rangle \neq 0\,, \qquad \langle 0|\psi_{mn} \neq 0\,, \qquad \text{iff } \begin{cases} m,n<0\\ m<0, n=0\,,\\ m=0, n<0\end{cases}\end{aligned} \tag{3.5}$$

it is compatible with the vacuum state of the bosonic fields. In fact we can show, from (3.2), that the relations

$$\lim_{z'\to z} :e^{iX(z,\bar z)/\sqrt 2}::e^{-iX(z',\bar z')/\sqrt 2}: = \frac{i}{\sqrt 2}\frac{\partial X(z)}{\partial z} + \frac{i}{\sqrt 2}\frac{\partial\tilde X(\bar z)}{\partial\bar z} + \text{const.} \tag{3.6}$$

are correct up to some (diverging) constants. Therefore a comparison of the coefficients of the power expansion in both sides yields

$$\begin{aligned}\alpha_{\pm n} &= \sqrt 2\sum_{r,s\in\mathbf Z}\psi_{r,s}\psi^*_{r\pm n,s}\,, \qquad n=0,1,2,3,\dots,\\ \tilde\alpha_{\pm n} &= \sqrt 2\sum_{r,s\in\mathbf Z}\psi_{r,s}\psi^*_{r,s\pm n}\,, \qquad n=0,1,2,3,\cdots,\\ \alpha_0 &= p\,, \qquad \tilde\alpha_0 = \tilde p\,.\end{aligned}$$

Hence we obtain

$$\begin{aligned}\alpha_n|0\rangle &= \tilde\alpha_n|0\rangle = 0\,, \qquad n\geq 0\,,\\ \langle 0|\alpha_{-n} &= \langle 0|\tilde\alpha_{-n} = 0\,, \qquad n>0\,,\end{aligned} \tag{3.7}$$

Now we want to derive a τ function associated with closed strings. The grassmannian is defined by means of fermion fields $\psi(z,\bar z)$ and $\psi^*(z,\bar z)$. The hamiltonian is not obvious in the present case, since a straightforward generalization of (2.7) does not make sense.

Instead we generalize (2.10) to

$$H(t,\tilde{t}) = \oint \frac{dz}{2\pi}\frac{\partial X(z)}{\partial z}\xi(t,z) + \oint \frac{d\bar{z}}{2\pi}\frac{\partial \tilde{X}(\bar{z})}{\partial \bar{z}}\xi(\tilde{t},\bar{z}), \tag{3.8}$$

from which follow

$$\begin{aligned} e^{H(t,\tilde{t})}\psi(z,\bar{z})e^{-H(t,\tilde{t})} &= e^{(\xi(t,z)+\xi(\tilde{t},\bar{z}))/\sqrt{2}}\psi(z,\bar{z})\,, \\ e^{H(t,\tilde{t})}\psi^*(z,\bar{z})e^{-H(t,\tilde{t})} &= e^{-(\xi(t,z)+\xi(\tilde{t},\bar{z}))/\sqrt{2}}\psi^*(z,\bar{z})\,. \end{aligned} \tag{3.9}$$

We then define the τ function by

$$\tau(t,\tilde{t}) = \langle 0|e^{H(t,\tilde{t})}G|0\rangle\,. \tag{3.10}$$

G in (3.10) is an element of the grassmannian of the fields $\psi(z,\bar{z})$ and $\psi^*(z,\bar{z})$. To be specific let V and V^* denote the linear space $\oplus \mathrm{C}\psi(z,\bar{z})$ and $\oplus \mathrm{C}\psi^*(z,\bar{z})$, respectively. Then $G \in G(V,V^*)$ satisfies

$$GV = VG\,, \qquad GV^* = V^*G\,.$$

In particular, the following relations must hold:

$$\begin{aligned} \psi(z,\bar{z})G &= \oint \frac{dz'}{2\pi i}\oint \frac{d\bar{z}'}{2\pi i}A(z,\bar{z};z',\bar{z}')G\psi(z',\bar{z}')\,, \\ G\psi^*(z,\bar{z}) &= \oint \frac{dz'}{2\pi i}\oint \frac{d\bar{z}'}{2\pi i}A(z',\bar{z}';z,\bar{z})\psi^*(z',\bar{z}')G\,. \end{aligned} \tag{3.11}$$

To see the correspondence between the τ function (3.10) and the closed string correlation function, we adopt the Miwa transformations in the form

$$\begin{aligned} \tilde{t}_0 &= -\sum_{j=1}^{M} k_j \ln \bar{z}_j\,, \\ \tilde{t}_n &= \frac{1}{n}\sum_{j=1}^{M} k_j \bar{z}_j^{-n}\,, \qquad n = 1,2,\dots\,, \end{aligned} \tag{3.12}$$

together with (2.11). The hamiltonian is expressed in terms of the closed string coordinate X

$$H(t,\tilde{t}) = i\sum_{j=1}^{M} k_j X_-(z_j,\bar{z}_j)\,. \tag{3.13}$$

For a given set of variables

$$K = (k_1,k_2,\dots,k_M,z_1,z_2,\dots,z_M,\bar{z}_1,\dots,\bar{z}_M) \tag{3.14}$$

we define the string correlation function F_G with the background G by

$$F_G(K) = \langle 0| : e^{ik_1X(z_1,\bar{z}_1)} : \cdots : e^{ik_MX(z_M,\bar{z}_M)} : G|0\rangle\,. \tag{3.15}$$

The correlation of higher spin particles are obtained from this expression via differentiation with respect to corresonding components of momenta k_j's.

By using the identity

$$\frac{:\phi_1::\phi_2:\cdots:\phi_N:}{\langle 0|:\phi_1::\phi_2:\cdots:\phi_N:|0\rangle} = :\phi_1\phi_2\cdots\phi_N:. \tag{3.16}$$

which holds for arbitrary fields ϕ_j's, the τ function (3.10) is then related to the correlation function by the following formula:

$$\tau(K) = \frac{F_G(K)}{F_1(K)}.$$

In the case of $G = \exp[a\psi(z,\bar z)\psi^*(z',\bar z')]$, for instance, we obtain the τ function describing the one soliton solution:

$$\tau_{1\mathrm{sol}} = 1 + \frac{a}{\sqrt{(z-z')(\bar z - \bar z')}} e^{\Xi(t,\tilde t,z,z')}, \tag{3.17}$$

where

$$\Xi(t,\tilde t,z,z') := \frac{\xi(t,z)-\xi(t,z')}{\sqrt{2}} + \frac{\xi(\tilde t,\bar z)-\xi(\tilde t,\bar z')}{\sqrt{2}}. \tag{3.18}$$

4. Conformally symmetric boundary states

We now turn to the problem of the conformally symmetric boundary states. In this section we study manifestly symmetric conformal boundary states. For this purpose we introduce the following notation

$$\Phi_\pm := \oint_B \frac{dz}{2\pi i}\oint_B \frac{dz'}{2\pi i}\frac{\partial\tilde X(1/z)}{\partial z}\Delta_\pm(z,z')\frac{\partial X(z')}{\partial z'}.$$

Here $\Delta_\pm(z,z')$ are those defined in (2.12), and the integrations are taken along the boundary B. We notice that they can be also expressed as integrations along certain space time closed paths as

$$\Phi_\pm := \oint \frac{d\tilde X(1/z)}{2\pi i}\oint \frac{dX(z')}{2\pi i}\Delta_\pm(z,z').$$

The Neumann boudary states are defined by

$$|N\rangle = e^{\Phi_+}|0\rangle, \qquad \langle N| = \langle 0|e^{\Phi_-}, \tag{4.1}$$

which are manifestly symmetric under the conformal transformations.

Owing to the relations

$$\left[X(z), \frac{\partial X(z')}{\partial z'}\right] = 2\pi i\delta(z-z'),$$
$$\left[\tilde X(z), \frac{\partial \tilde X(z')}{\partial z'}\right] = 2\pi i\delta(z-z')$$

we can verify

$$
\begin{aligned}
[\Phi_\pm, X(y)] &= \begin{pmatrix} -\tilde{X}_+(1/y) + i\tilde{p}\ln|z| + \tilde{x}_0 \\ \tilde{X}_-(1/y) \end{pmatrix} \\
[\Phi_\pm, \tilde{X}(y)] &= \begin{pmatrix} -X_+(1/y) + ip\ln|y| + x_0 \\ X_-(1/y) - ip\ln|yz'| \end{pmatrix} \\
[\Phi_\pm, [\Phi_\pm, X(y)]] &= [\Phi_\pm, [\Phi_\pm, \tilde{X}(y)]] = 0\,.
\end{aligned}
$$

We are thus convinced the following relations

$$
X(y)|N\rangle = \tilde{X}\left(\frac{1}{y}\right)|N\rangle\,, \qquad \langle N|X(y) = \langle N|\tilde{X}\left(\frac{1}{y}\right). \tag{4.2}
$$

The meaning of the formulae of (4.2) is that when a right moving field reflects at the boundary it turns to a left moving one. In particular we have

$$
: e^{\pm iX(z,\bar{z})/\sqrt{2}} : |N\rangle =: e^{\pm i\sqrt{2}X(z)} : |N\rangle = J^\pm(z)|N\rangle
$$

when $\bar{z}z = 1$. The result by Callan et al. [1], that the dynamical boundary state $|B\rangle$ is obtained from $|N\rangle$ by the SU(2) rotation, owes to this fact.

Following to the argument of [1] let us further consider the scattering of a field $\partial\tilde{X}/\partial\bar{z}$ at the boundary. It is more convenient to introduce $\tilde{J}^3(\bar{z})$ by

$$
\tilde{J}^3(\bar{z}) = \frac{i}{\sqrt{2}}\frac{\partial\tilde{X}(\bar{z})}{\partial\bar{z}}\,. \tag{4.3}
$$

Since $\tilde{J}^3$ commutes with $J^\pm$ it hits directly to the Neumann boundary state $\langle N|$ when it acts on $\langle B|$. At the boundary it turns to a holomorphic operator $J^3(z)$ and then undergoes an SU(2) rotation before it is reflected. Namely

$$
\begin{aligned}
\langle B|\tilde{J}^3(\bar{z}) &= \langle N|J^3(z)e^{-i\theta_a J^a} \\
&= \langle B|J^\theta(z)\,,
\end{aligned} \tag{4.4}
$$

where

$$
\begin{aligned}
J^\theta(z) &:= e^{i\theta_a J^a}J^3(z)e^{-i\theta_a J^a} \\
&= \cos(2\pi|g|)J^3(z) + \sin(2\pi|g|)\frac{gJ^+(z) - \bar{g}J^-(z)}{2i|g|}\,.
\end{aligned} \tag{4.5}
$$

Apart from the particle sector represented by the holomorphic field $\partial X/\partial z$ there appear other fields $\exp[\pm i\sqrt{2}X(z)]$ if $g \neq 0$. The new sectors created by the reflection were interpreted as a production of charged solitons in [1].

5. Soliton equations satisfied by closed string correlation functions

The purpose of this section is to derive a soliton equation satisfied by the closed string correlation functions with conformally symmetric boundaries. It will be done by generalizing the correlation function (3.15) to the case with the boundary:

$$
F_G^B(K) = \langle B| : e^{ik_1X(z_1,\bar{z}_1)} : \cdots : e^{ik_MX(z_M,\bar{z}_M)} : G|0\rangle\,. \tag{5.1}
$$

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This is also a generalization of the formulation of [1] to the case with a background specified by the universal grassmannian G.

We want to derive a bilinear relation satisfied by the correlation function (5.1). To this end we first consider the following integration:

$$I = \oint_0 \frac{dz}{2\pi i} \oint_\infty \frac{d\bar z}{2\pi i} \psi^*(z,\bar z) G|0\rangle \otimes \psi(z,\bar z) G|0\rangle \,,$$

where we assume that the contours encircle around $z=0$ and $\bar z = \infty$. Applying the relations (3.11) this is equivalent to

$$= \oint_0 \frac{dz}{2\pi i} \oint_\infty \frac{d\bar z}{2\pi i} G\psi^*(z,\bar z)|0\rangle \otimes G\psi(z,\bar z)|0\rangle \,,$$

which turns out to be zero, i.e.,

$$I = 0 \,, \tag{5.2}$$

owing to the fact that $\psi^*(z,\bar z)|0\rangle$ and $\psi(z,\bar z)|0\rangle$ are orthogonal.

This fundamental property will be verified by rewriting I as

$$I = \sum_{p,q\in\mathbb{Z}} G\psi^*_{p,q}|0\rangle \otimes G\psi_{p,q}|0\rangle \,.$$

Each term of the summation is identically zero due to the nature of the vacuum state (3.5).

We can interprete the formula (5.2) as a bilinear sum rule for the correlation functions (5.1) if we multiply to I the state

$$\begin{aligned}&\langle B| :e^{ik_1X(z_1,\bar z_1)}::e^{ik_2X(z_2,\bar z_2)}:\cdots:e^{ik_MX(z_M,\bar z_M)}:\otimes\\ &\otimes\langle B| :e^{ik'_1X(z_1,\bar z_1)}::e^{ik'_2X(z_2,\bar z_2)}:\cdots:e^{ik'_MX(z_M,\bar z_M)}:\end{aligned}$$

i.e.,

$$\oint_0 \frac{dz}{2\pi i} \oint_\infty \frac{d\bar z}{2\pi i} F^B_{\psi^*(z,\bar z)G}(K) F^B_{\psi(z,\bar z)G}(K') = 0. \tag{5.3}$$

This is a formula every closed string correlation function must satisfy.

In order to find more useful informations we divide (5.3) by $F^B_1(K)F^B_1(K')$ and rewrite it as

$$\oint_0 \frac{dz}{2\pi i} \oint_\infty \frac{d\bar z}{2\pi i} \langle B| :e^{i\sum_j k_jX(z_j,\bar z_j)}: \psi^*(z,\bar z)G|0\rangle \langle B| :e^{i\sum_j k'_jX(z_j,\bar z_j)}: \psi(z,\bar z)G|0\rangle = 0 \,,$$

where we used (3.16).

Now let us recall the expressions (3.2) of ψ and ψ^* and shift their plus components $X_+(z,\bar z)$ to the left to obtain

$$\begin{aligned}\oint_0 \frac{dz}{2\pi i} \oint_\infty \frac{d\bar z}{2\pi i} \prod_{j=1}^{M} ((z_j - z)(\bar z_j - \bar z))^{(k'_j-k_j)/\sqrt2} \times&\\ \times\langle B| :e^{i\sum_j k_jX(z_j,\bar z_j) - iX(z,\bar z)/\sqrt2}: G|0\rangle\times&\\ \times\langle B| :e^{i\sum_j k'_jX(z_j,\bar z_j) + iX(z,\bar z)/\sqrt2}: G|0\rangle = 0 \,.&\end{aligned} \tag{5.4}$$

So far we have not specified k_j's and k'_j's. Since they are arbitrary we can choose

$$k'_j = k_j - \sqrt{2} \qquad (j = 1, 2, 3)\,, \qquad k'_j = k_j \qquad (j \neq 1, 2, 3) \tag{5.5}$$

such that the factor

$$\prod_{j=1}^{M} \left((z_j - z)(\bar{z}_j - \bar{z})\right)^{(k'_j - k_j)/\sqrt{2}}$$

in (5.4) becomes simply

$$\prod_{j=1}^{3} \frac{1}{(z - z_j)(\bar{z} - \bar{z}_j)}\,. \tag{5.6}$$

We are ready to move the contours of integrations in (5.4) toward the boundary. We assume that all of the simple poles of (5.6) on the z plane are in between the boundary and the origins and those on the $\bar{z}$ plane are in between the boundary and ∞. We notice that, while the contours are moved toward the boundary, they are only singularities which contribute to the integrations. Therefore we obtain the bilinear relation for the τ function with boundary, which is defined by

$$\tau^B(k_1, k_2, k_3) = \frac{F^B_G(K)}{F^B_1(K)}\,,$$

as follows:

$$\begin{aligned}
&\frac{\tau^B(k_1 + \sqrt{2}, k_2, k_3)\tau^B(k_1, k_2 + \sqrt{2}, k_3 + \sqrt{2})}{(z_1 - z_2)(z_1 - z_3)} +\\
&+ \frac{\tau^B(k_1, k_2 + \sqrt{2}, k_3)\tau^B(k_1 + \sqrt{2}, k_2, k_3 + \sqrt{2})}{(z_2 - z_1)(z_2 - z_3)} +\\
&+ \frac{\tau^B(k_1, k_2, k_3 + \sqrt{2})\tau^B(k_1 + \sqrt{2}, k_2 + \sqrt{2}, k_3)}{(z_3 - z_1)(z_3 - z_2)} +\\
&+ \oint_B \frac{dz}{2\pi i} \oint_B \frac{d\bar{z}}{2\pi i} \prod_{j=1}^{3} \frac{1}{(z - z_j)(\bar{z} - \bar{z}_j)} \times\\
&\times \tau^B\left(k_1, k_2, k_3, \frac{-1}{\sqrt{2}}\right) \tau^B\left(k'_1, k'_2, k'_3, \frac{1}{\sqrt{2}}\right) = 0\,.
\end{aligned} \tag{5.7}$$

In this expression $k'_j = k_j + 1/\sqrt{2}$.

A few comments are in order.

1. If some of z_j's and $\bar{z}_l^{-1}$'s are not in between the origins and the boundary, they do not contribute to the summation of (5.7).

2. If there is no boundary and the contours of the integrations are moved away to infinity the second term of (5.7) does not contribute and we obtain a generalization of HBDE (2.18) to the closed strings.

3. If we are concerned only to deriving bilinear identities of the τ function we could generalize the Miwa transformation (3.12) to

$$\tilde{t}_0 = -\sum_{j=1}^{M} \tilde{k}_j \ln \bar{z}_j \,,$$
$$\tilde{t}_n = \frac{1}{n}\sum_{j=1}^{M} \tilde{k}_j \bar{z}_j^{-n} \,, \qquad n = 1, 2, \dots \,,$$

such that $\tilde{k}_j$'s are independent from k_j's.

6. Correspondence of solitons in two theories

In the paper [1] by Callan et al. it was argued that the field X, which was at a minimum of the potential, is displaced by an integer number of period $2\pi\sqrt{2}$ after scattering from the boundary. The shift is visible as a topological soliton which is propagating away from the boundary. The generators, which carry the topological numbers, are identified with the weight one field $e^{\pm i\sqrt{2}X(z)}$. The exponents describe kinks with a shift of $2\pi\sqrt{2}$ between values of X before and after the scattering.

On the other hand we have derived in this paper soliton equations which are satisfied by the correlation functions of closed strings. The question is whether solitons which appear in two theories have some relation? The purpose of this section is to clarify this point.

In the theory of KP-hierarchy[9] a creation of one soliton state is generated by the operation of

$$e^{a\Lambda(z)\Lambda^*(z')} \,, \tag{6.1}$$

where

$$\Lambda(z) = e^{\xi(t,z)} e^{\xi(\partial_t, z^{-1})} \,, \qquad \Lambda^*(z) = e^{-\xi(t,z)} e^{-\xi(\partial_t, z^{-1})}$$

to the τ function. Here ∂_t means $(\partial_{t_1}, \partial_{t_2}, \partial_{t_3}, \cdots)$. One soliton solution (2.6), for example, will be seen being generated from $\langle 0| e^{H(t)} |0\rangle = 1$ by this operation. If we expand $\Lambda(z)\Lambda^*(z')$ as

$$\Lambda(z)\Lambda^*(z') = \sum_{m,n\in\mathbb{Z}} L_{mn} z^m z'^{-n}$$

the coefficients satisfy

$$[L_{mn}, L_{m'n'}] = \delta_{m'n} L_{mn'} - \delta_{mn'} L_{m'n} \,.$$

Hence (6.1) is an element of the group $GL(\infty)$.

We can translate this symmetry into a simple word of the language of string theory. Namely the actions of $\Lambda(z)$ and $\Lambda^*(z)$ are equivalent to the shifting $kX(z)$ to $(k \pm 1)X(z)$, which we can express as

$$\Lambda(z) = e^{\partial_k} \,, \qquad \Lambda^*(z) = e^{-\partial_k} \,.$$

If z coincides with one of the z_j's in the hamiltonian (2.13), $\Lambda(z_j)$ increases the corresponding value of k_j by one. If z coincides with none of them, it creates a new field $e^{iX(z)}$ in the string correlation functions. The generation of a soliton in the soliton theory, which is achieved by an action of the operator (6.1), is equivalent to an action of

$$1 + ae^{\partial_k} e^{-\partial_{k'}}$$

to the string correlation functions. It is apparent that the action of the second term adds a pair of fields $: e^{iX(z)} :: e^{-iX(z')} :$ to the correlation functions. Since this process generates new solutions to the HBDE (2.18) it is a Bäcklund transformation [10].

Let us extend above consideration to the closed strings. Instead of $\psi(z)$, $\psi^*(z)$ in (2.6) a soliton state will be described by the fields $\psi(z, \bar{z})$ and $\psi^*(z, \bar{z})$. The bosonization procedure (3.2) enables us to express them in terms of string coordinates. The corresponding generator of the Bäcklund transformation is

$$\Lambda(z, \bar{z})\Lambda^*(z', \bar{z}') = e^{(1/\sqrt{2})\partial_k} e^{-(1/\sqrt{2})\partial_{k'}} ,$$

which introduces a pair of fields

$$\psi(z, \bar{z})\psi^*(z', \bar{z}') =: e^{iX(z,\bar{z})/\sqrt{2}} :: e^{-iX(z',\bar{z}')/\sqrt{2}} : \tag{6.2}$$

into the string correlation functions. We will call this field (6.2) a 'classical soliton field'.

As it was discussed in [1] the pair of fields of (6.2) is equivalent to

$$: e^{i\sqrt{2}X(z)} :: e^{-i\sqrt{2}X(z')} :$$

when it is on the boundary. Let us call $e^{\pm i\sqrt{2}X(z)}$ the 'BCFT soliton fields'. From this observation we find that a pair of 'BCFT soliton fields' in [1] form a 'classical soliton field' of classical soliton equations. In other words a soliton of classical equations is a neutral bound state of a pair of charged fields in the boundary CFT.

The profile of the soliton solution to classical equations has been represented by (2.1). It is an exponential function of the soliton coordinates t_n's. The special dependence on t_n's implies the solitary nature of the solution when the Toda amplitude r_m is expressed in terms of the τ function via (2.3). The question is what is the meaning of the classical *soliton coordinates* t_n in the string theory?

In order to answer this question we rewrite the hamiltonian $H(t, \tilde{t})$ of (3.8), which generates the motion of classical solitons, as

$$\begin{aligned} H(t, \tilde{t}) &= i \sum_j k_j X_-(z_j, \bar{z}_j) \\ &= -\sum_{n=0}^{\infty} (t_n \alpha_n - \tilde{t}_n \tilde{\alpha}_n) . \end{aligned} \tag{6.3}$$

We notice that α_n's and $\tilde{\alpha}_n$'s are particle modes of a string. Therefore the two expressions in (6.3) present two different pictures, i.e. a string picture and a particle picture. This also provides a physical interpretation of the Miwa transformation (2.11).

The direction of motion of a classical soliton depends on which particle mode of the string is excited. Moreover (3.9) shows that the velocity of the classical soliton moving along t_n is determined by the eigenvalue z^n of α_n as it was the case of the Toda lattice (2.4).

7. One soliton solution

To be specific let us study the correlation function of closed strings (5.1) with the boundary state $\langle B|$ and G is given by $\exp[a\psi(z,\bar{z})\psi^*(z',\bar{z}')]$. If we restrict the values of k_j's to multiple of $1/\sqrt{2}$ it is a correlation function of 'BCFT soliton fields'. The presence of G amounts to introduce a 'classical soliton field' into the correlation function. But, once the pair is substituted into the function, it is not distinguished from other 'BCFT soliton fields'.

In the language of τ functions we have

$$\begin{aligned}
\tau^B_{1\mathrm{sol}} &= 1 + a\frac{F^B_{\psi(z,\bar{z})\psi^*(z',\bar{z}')}(K)}{F^B_1(K)} \\
&= 1 + a\frac{\langle B| : e^{i\sum_j k_j X(z_j,\bar{z}_j)} : \psi(z,\bar{z})\psi^*(z',\bar{z}')|0\rangle}{\langle B| : e^{i\sum_j k_j X(z_j,\bar{z}_j)} : |0\rangle} \\
&= 1 + a\frac{W(t,\tilde{t})}{\sqrt{(z-z')(\bar{z}-\bar{z}')}} e^{\Xi(t,\tilde{t},z,z')} ,
\end{aligned}$$

where Ξ is given by (3.18) and we used (3.9) to obtain the exponential factor. The existence of other 'BCFT soliton fields' causes an effect on the 'classical soliton fields' $\psi(z,\bar{z})\psi^*(z',\bar{z}')$ to move along the direction determined by the ξ's in this exponential factor.

Comparing with the one soliton state (3.17) with no boundary, there is an extra factor W which now depends on t's and is given by

$$W = \frac{\langle B| e^{i\sum_j k_j X_+(z_j,\bar{z}_j)} e^{i(X_+(z,\bar{z}) - X_+(z',\bar{z}'))/\sqrt{2}}|0\rangle}{\langle B| e^{i\sum_j k_j X_+(z_j,\bar{z}_j)}|0\rangle} .$$

Let us compute this quantity to see its t dependence. We first bring all $\tilde{X}_+$ components to the left untill they hit to the boundary state $\langle N|$. After all of the right moving components are reflected into the left moving ones X_-, the state $\langle N|$ will be turned into the vacuum state $\langle 0|$. Hence the rest we have to calculate is

$$\frac{\langle 0| e^{i\sum_{j=1}^{M+2} k_j X_-(\bar{z}_j^{-1})} e^{-i\theta_a J^a} e^{i\sum_{j=1}^{M+2} k_j X_+(z_j)}|0\rangle}{\langle 0| e^{i\sum_{j=1}^{M} k_j X_-(\bar{z}_j^{-1})} e^{-i\theta_a J^a} e^{i\sum_{j=1}^{M} k_j X_+(z_j)}|0\rangle} ,$$

where we used the notations

$$k_{M+1} = -k_{M+2} = \frac{1}{\sqrt{2}} , \qquad z_{M+1} = z , \qquad z_{M+2} = z' .$$

Applying the formula (4.5), we obtain

$$W = \left(\frac{(\bar{z}^{-1} - z)(\bar{z}'^{-1} - z')}{(\bar{z}'^{-1} - z)(\bar{z}^{-1} - z')}\right)^{\frac{1}{2}\cos(2\pi|g|)} \prod_{j=1}^{M} \left(\frac{(\bar{z}^{-1} - z_j)(\bar{z}_j^{-1} - z)}{(\bar{z}'^{-1} - z_j)(\bar{z}_j^{-1} - z')}\right)^{\frac{1}{\sqrt{2}}k_j\cos(2\pi|g|)} .$$

Under the conditions that all z_j's are on the unit circle and $\sum_j k_j = 0$ we can write the second factor of W as

$$e^{\Xi(t,\tilde{t},z,z')\cos(2\pi|g|)} .$$

If we combine these results together the modification to the τ function due to the boundary is rather simple

$$\tau^B_{1\mathrm{sol}} = 1 + a\frac{y^{\cos^2(\pi|g|)}}{\sqrt{(z-z')(\bar{z}-\bar{z}')y}}e^{\Xi(t,\tilde{t},z,z')2\cos^2(\pi|g|)}\,,$$

where y is the cross ratio defined by

$$y := \frac{(\bar{z}^{-1}-z)(\bar{z}'^{-1}-z')}{(\bar{z}'^{-1}-z)(\bar{z}^{-1}-z')}\,.$$

Let us see how the profile of the soliton solution of the Toda equation will be affected by a change of the boundary parameter g. Comparing with the data (2.4) we find

$$e^{r_m} - 1 = \frac{\omega'^2}{\cosh^2(\kappa' m + \omega' t + \delta')}$$

where

$$\begin{aligned}\kappa' &= 2\sqrt{2}\kappa\cos^2(\pi|g|)\,, \qquad \omega' = 2\sqrt{2}\omega\cos^2(\pi|g|)\,,\\ \delta' &= \delta + \frac{1}{4}\cos(2\pi|g|)\ln y\end{aligned}$$

and assumed z and z' being real.

$$t := \frac{t_3+\tilde{t}_3}{2}, \qquad m := \frac{t_1+\tilde{t}_1}{2}\,.$$

An interesting feature of this result is that the soliton is completely frozen when $|g|$ is a half integer corresponding to the Dirichlet boundary condition in the string picture. We have also calculated two soliton solution influenced by the boundary and found that they are stable under the collisions. Details will be reported elsewhere.

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References

[1] C.G. Callan Jr., I.R. Klebanov, A.W.W. Ludwig and J.M. Maldacena, *Exact solution of a boundary conformal field theory*, *Nucl. Phys.* **B 422** (1994) 417 [hep-th/9402113].

[2] S. Saito, *String theories and Hirota's bilinear difference equation*, *Phys. Rev. Lett.* **59** (1987) 1798; *String vertex on the grassmannian*, *Phys. Rev.* **D 37** (1988) 990;
K. Sogo, *A way from string to soliton - Introduction of KP coordinate to string amplitudes*, *J. Phys. Soc. Jpn.* **56** (1987) 2291.

[3] R. Hirota, *Discrete analogue of a generalized Toda equation*, *J. Phys. Soc. Jpn.* **50** (1981) 3785.

[4] T. Miwa, *On Hirota's difference equations*, *Proc. Japan Acad. A* **58** (1982) 9.

[5] M. Sato and Y. Sato, *Publ. RIMS* **388** (1980) 183;
M. Sato and Y. Sato, *Publ. RIMS* **414** (1981) 181;
M. Sato, *Publ. RIMS* **433** (1981) 3.

[6] H. Kato and S. Saito, *Off-shell strings and generalization of Fay's addition theorem*, *Lett. Math. Phys.* **18** (1989) 177.

[7] D. Mumford, *Tata lectures on theta I, II*, Birkhäuser, 1983.

[8] J.D. Fay, *Theta functions on Riemann surfaces*, Lect. Notes in Math. 352, Springe-Verlag, 1973.

[9] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, *Operator approach to the Kadomtsev-Petviashvili equation - Transformation groups for soliton equations III, VI*, *J. Phys. Soc. Jpn.* **50** (1981) 3806.

[10] N. Saitoh and S. Saito, *Integrable lattice gauge model based on soliton theory*, *J. Phys.* **A 23** (1990) 3017.